Lattices & Factoring

Invited Talk

Léo Ducas

CWI, Amsterdam, The Netherlands

PKC, May 10th, 2021

Modern Cryptography, This Old Thing

Cryptography is getting old.

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Cryptography has reached a non-negligible age.

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Cryptography has reached a non-negligible age.

Let's write our history before it gets lost.

Knapsack-based Cryptography

Typical narrative on Knapsack-based cryptography

- An embarrassement to forget
- Ajtai single-handedly put an end to that dark Era

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> If I have seen further,
> it is by standing on the shoulders of Giants.
>
> – *Isaac Newton*

Ajtai is a Giant of Lattice-based Cryptography.
Let's enjoy the the view he had from the shoulders his own Giants.

Today's Giants

Factoring with Lattices Short Vectors

C-P. Schnorr

L. Adleman

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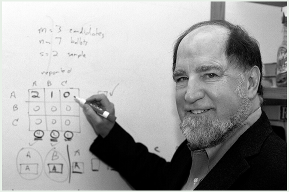
Decoding Lattices by Factorization

B. Chor

R. Rivest



Part I:
Factoring with Lattice Short Vectors

Factoring: the Quadratic Sieve [Pomerance, 1981]

Notation : $\equiv$ for congruence modulo N

Goal: Find a non-trivial[1] solution to $X^2 \equiv Y^2$

$\Rightarrow (X - Y)(X + Y) \equiv 0$

$\Rightarrow \gcd(X \pm Y, N)$ is a non-trivial factor of N

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A two-steps process:

- Collect Relations
- Linear Algebra

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Step 1: Relation Collection

- Define a **factor basis**: $\mathcal{F} = \{p | p \text{ is primes}, p \leq B\}$
- Repeat:

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- Repeat:
  - Pick random X, compute $Z = X^2 \bmod N$
  - Use **trial division** to write $Z = \prod p_i^{e_i}$ $(p_i \in \mathcal{F})$
  - If successful, store the **relation** $X^2 \equiv \prod p_i^{e_i}$
- Until B relations are collected

The complexity trade-off

- Increasing B improves the success probability of each trial
- But more relations are needed
- The optimum is at $B = \exp(\tilde{O}(\sqrt{\log N}))$ $= L_N(1/2)$

Step 2: Linear Algebra

- We have collected relations:

$$\begin{array}{cccccc}
X_1^2 & \equiv & {p_1}^{e_{1,1}} & {p_2}^{e_{1,2}} & {p_3}^{e_{1,3}} & \cdots \\
X_2^2 & \equiv & {p_1}^{e_{2,1}} & {p_2}^{e_{2,2}} & {p_3}^{e_{2,3}} & \cdots \\
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- Combine the above to make all exponents even integers

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- Obtain a solution to

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Could we aim for $X^2 \bmod N$ that are significantly smaller ?

Choose $X \approx \sqrt{N}$, so that $X^2 \approx N$

If $X = \sqrt{N} + \epsilon$, with $\epsilon \ll \sqrt{N}$, then:

$$X^2 \equiv 2\epsilon\sqrt{N} + \epsilon^2$$

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The complexity gain

Improves the hidden constant in $\exp(\tilde{O}(\sqrt{\log N})) \qquad = L_N(1/2)$

Aiming Better [Schnorr 1991]

A Relaxation

The left-hand-side needs not be square, B-smooth can do as well:

$$p_1{}^{e'_1} p_2{}^{e'_2} p_3{}^{e'_3} \cdots \equiv p_1{}^{e_1} p_2{}^{e_2} p_3{}^{e_3} \cdots$$

$$1 \equiv p_1{}^{e_1 - e'_1} p_2{}^{e_2 - e'_2} p_3{}^{e_3 - e'_3} \ldots$$

Our New Goal

Find positive exponents $(e'_1, e'_2, e'_3, \ldots)$ such that

$$p_1{}^{e'_1} p_2{}^{e'_2} p_3{}^{e'_3} \cdots \approx N$$

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This is an (approximate) knapsack problem !

$$e'_1 \ln p_1 + e'_2 \ln p_2 + e'_3 \ln p_3 + \cdots \approx \ln N$$

Aiming with lattices [Schnorr 1991]

Choose a constant C to rewrite the knapsack as a lattice CVP

$$\begin{bmatrix} \ln p_1 & & & & \\ & \ln p_2 & & & \\ & & \ln p_3 & & \\ & & & \ddots & \\ & & & & \ln p_n \\ C\ln p_1 & C\ln p_2 & C\ln p_3 & \cdots & C\ln p_n \end{bmatrix} \cdot \begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \\ \vdots \\ e'_n \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ C\ln N \end{bmatrix}$$

Knapsack $\neq$ CVP

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But that might be OK !

- $u/v \approx N \Rightarrow u \approx vN$, therefore $S = u - vN$ may be small
- Quality degrades as $v = \prod_{e'_i<0} p_i^{-e_i}$ gets larger

Attempting Average-Case Analysis

Lattice Pitfalls

- The lattice is not full dimensional — apply due projections
- Gaussian Heuristic seems invalid — for certain C
- The ℓ_2 norm is a bit inadequate — ℓ_1 more relevant
- Naive predictions of ℓ_2/ℓ_1 can also fail

Trial Division Pitfall

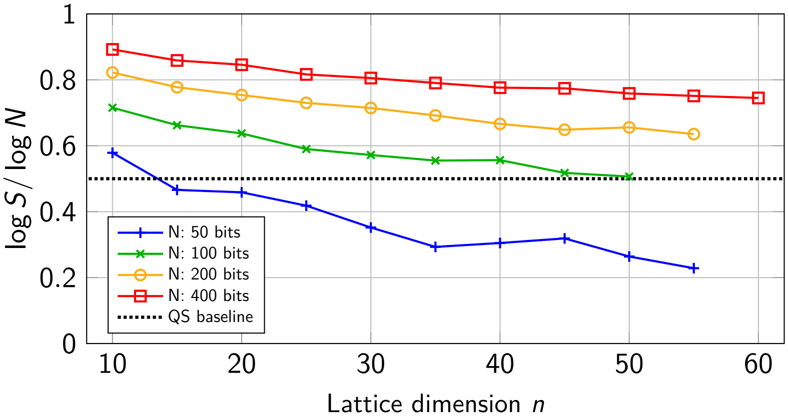
- B-Smoothness probability of $S = u - vN$ lower than expected

$$p_i | u \vee p_i | v \Rightarrow p_i \not| S$$

Mind the Variants

- Most papers force $B = p_n$ or $B = 1$. Here: B unconstrained.
- The diagonal part of the lattice may vary as well.

Experiments



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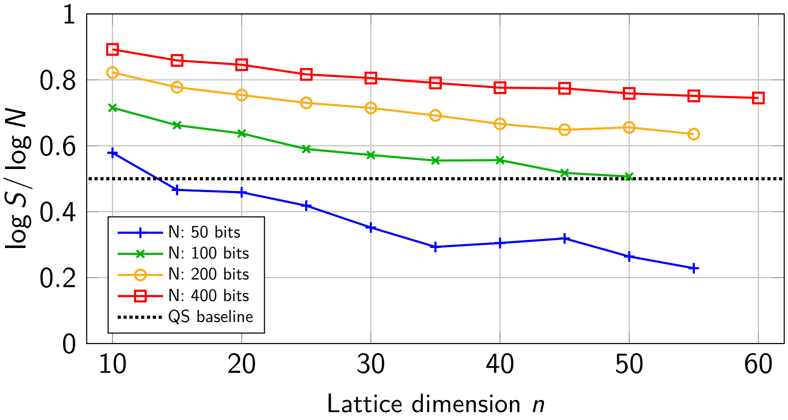

The size of S roughly dictates the cost of the non-lattice steps

For factoring a 100-bits N, to beat QS at the non-lattice steps, we should need a lattice dimension of at least $n \geq 50$.

My two Cents

- It's a deep and brilliant idea . . . that doesn't seem to work ☹
- A solid complexity analysis is still missing

  and appears quite challenging . . .
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  - An attempt at proving SVP $\geq$ Factoring [Adleman 1995]
  - A sucessful proof of NP-hardness for SVP [Ajtai 1998]
  - Idea reused for in relation to the abc-conjecture [Bright 2014]
  - Idea reused in a Module-LLL Algorithm [LPSW 2019]

A Surprising Twist [Ajtai 1998]

Recall the gap between Knapsack and SVP

- Knapsack solutions $\in \{0,1\}^n$, SVP solution $\mathbb{Z}^n$
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The key Insight

- $\{0,1\}^n$ solutions in Schnorr-Adleman lattice are in correspondence with smooth and square-free integers
- We know how to count those !

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A proof that SVP $\geq$ Knapsack

- Therefore SVP is NP-hard
- Learn more from Daniele's talk next week at the RISC seminar

Part II:
Decoding Lattices by Factorization

Dense Lattice with Efficient Decoding

In this whole section we work with the ℓ_1 norm !

Bounded Distance Decoding with radius r

- Given $t = v + e$ where $v \in \mathcal{L}$ and $\|e\| \leq r$
- Recover v and e

Unique solution guaranteed for $r \leq \lambda_1(\mathcal{L})/2$.

Minkowsky's bound

$$\frac{\lambda_1(\mathcal{L})}{\det(\mathcal{L})^{1/n}} \leq O(n)$$

We want a lattice and decoding alg. close to this bound.

Chor-Rivest Cryptosystem and Friends

The Key Idea [Chor Rivest 1988]

- Subset-sums is hard
- Subset-product is easy (trial divisions)
- Take logarithm, disguise the later as the former, get crypto.

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A Coding Gem Hidden Inside

- [Brier *et al.* '15]: Remove crypto from [NS'97], hides a good decodable binary code.
- [D. Pierrot '18]: [CR88, OTU00], hides a good decodable lattice.

Chor-Rivest Lattice (over the integers)

- Choose a modulus $M = 3^k$
- And a factor basis $\mathcal{F} = \{2, 5, 7, 11, 13, \ldots, p_n\}$

$B := p_n \sim n \ln n$

- Define the morphism $\psi : \mathbb{Z}^n \to (\mathbb{Z}/M\mathbb{Z})^*$:

$$\psi : x \mapsto \prod p_i^{x_i} \bmod M$$

- And finally define the kernel lattice

$$\mathcal{L} := \ker \psi = \left\{ v \in \mathbb{Z}^n \mid \prod p_i^{v_i} = 1 \bmod M \right\}$$

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The lattice can be computed efficiently !

- Discrete logarithms modulo $M = 3^k$ is easy
- Rewrites as a subset-sum lattice

$$\mathcal{L} = \left\{ v \in \mathbb{Z}^n \mid \sum v_i \operatorname{dlog} p_i = 0 \bmod \varphi(M) \right\}$$

Lattice Parameters

Lattice parameters

- $\dim \mathcal{L} = n$
- $\det \mathcal{L} \leq \varphi(M) \leq M$

Claim: $\lambda_1(\mathcal{L}) \geq \log M / \log B$ (Not exactly true ...)

- Recall that $\mathcal{L} = \{v \in \mathbb{Z}^n \mid \prod p_i^{v_i} = 1 \mod M\}$.
- For $v \neq 0$ to be in $\mathcal{L}$, $\prod p_i^{v_i}$ must wrap around $\mod M$
- In particular $B^{\|v\|_1} \geq M$ (This proof is a bit bogus !)

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Instantiate with $k = n$, i.e. $M = 3^n$

$$\frac{\lambda_1(\mathcal{L})}{\det(\mathcal{L})^{1/n}} \geq O\left(\frac{n}{\log n}\right)$$

That is only $O(\log n)$ factor away from Minkowsky bound.

Decoding Chor-Rivest Lattice

Bounded Distance Decoding with radius $r = \log M / \log B$

- Given $t = v + e$ where $v \in \mathcal{L}$ and $\|e\| \leq r$
- Recover v and e

- Compute

$$f = \prod p_i^{t_i} \bmod M = \prod p_i^{v_i} \prod p_i^{e_i} \bmod M$$
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- Note $\prod p_i^{e_i} \leq B^r \leq M$: we know it over $\mathbb{Z}$ not just $\bmod M$
- Factorize it by trial division: recover e

Dealing with Negative Errors

Now assume $2 \cdot B^r < \sqrt{M}$.

$$f = \prod_{e_i > 0}^{n} p_i^{e_i} \cdot \prod_{e_i < 0} p_i^{e_i} = u/v \mod M.$$

Lemma (Recovering u, v given f and M)

Let u, v, M be coprime s.t. $u, v < \sqrt{M/2}$, and let $f = u/v \bmod M$. Then, $\pm(u, v)$ are the shortest vectors of the 2-dimensional lattice

$$L = \{(x, y) \in \mathbb{Z}^2 | x - fy = 0 \bmod M\}.$$

In particular, given f and M, one can recover (u, v) in poly-time.

The last mile ?

We are still $O(\log n)$ away from Minkowsky's bound...
The issue is that we do not have enough small primes.
To get down to $O(1)$ away from Minkowsky's bound, we need

$$n \text{ primes of 'size' } O(1).$$

- Switching back from $\mathbb{Z}$ to $\mathbb{F}_p[X]$ doesn't improve asymptotics
- Elliptic curves could ?
- And what about Mordell-Weil lattices ? [Shioda '91, Elkies '94]

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A Recent Result

Using a completely different approach (construction D lattice over BCH codes), we are now $O(\sqrt{\log n})$ away from Minkowsky's bound [Mook Peikert 2020]

Why Cryptographers Should Care

Chor-Rivest Knapsack Cryptosystem is *not* Broken

- And offers very short ciphertexts !
- The underlying assumption is intriguing, especially quantumly

  Some kind of reverse of discrete logarithm problem

Chor-Rivest Decoding can be practical [Li Ling Xing Yeo 2017]

- Better decoding in a pure LWE-based scheme ?

And for Something Completely Different [Galbraith Li 2020]

- VBB Obfuscation of "near-equality" tests !

Part III:
A Critique of Research in Lattice-Based Cryptography

The SIS/LWE Monoculture[2]

Due credits

SIS/LWE formalism have achieved impressive feats, and the foundational work from TCS experts was exceptionally thorough.

[2] Not a critique of the contributions, but of what we have done of them.

The SIS/LWE Monoculture[2]

Due credits

SIS/LWE formalism have achieved impressive feats, and the foundational work from TCS experts was exceptionally thorough.

But . . .

- Worst-case hardness is not a silver bullet
  and does not dispense us from cryptanalysis
- We have locked ourselves in subspace of designs
  and current designs likely far from optimal
- Some very interesting ideas have been buried
  if not demoted to cryptographic sins

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A Diversity of Lattices

$\mathbb{Z}^n$, the Saddest of all Lattices

All algorithmic tasks (encode, decode, sample) in lattice-based cryptography are reduced to $\mathbb{Z}$ or $\mathbb{Z}^n$.
Yet, geometrically (packing, covering, ...) $\mathbb{Z}^n$ is the **worst** lattice.

[3] If you ever deal with prime cyclotomics rings, please read https://www.math.leidenuniv.nl/scripties/BachVanWoerden.pdf

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There are so many more !

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Root lattices[3] , Leech lattice, Construction D lattices, Barnes-Well lattices, Craig's lattices, Schnorr-Adleman lattices, Chor-Rivest lattices, Mordell-Weil lattices, ...

http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/

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Cryptography Strives in Diversity !

Lattice-based Cryptography needs:

- More diversity of Backgrounds
- More diversity of Point of View
- More diversity of Goals
- More diversity of **People** !

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Thank You !